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Rational Chebyshev Approximations for the Bickley Functions $Ki_n(x)$

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Abstract. This report presents near-minimax rational approximations for the Bickley functions $Ki_1(x)$ for $x \geqslant 0$, $Ki_2(x)$ and $Ki_3(x)$ for $0 \leqslant x \leqslant 6$, and $Ki_8(x)$, $Ki_9(x)$ and $Ki_{10}(x)$ for $x \geqslant 6$, with relative errors ranging down to 10^{-23}. The approximations, combined with the recurrence relation, yield a stable method of computing $Ki_n(x)$, $n = 1, 2, \ldots, 10$, for the complete range of the argument.

1. Introduction. The Bessel function integrals defined by

$$Ki_n(x) = \int_x^\infty Ki_{n-1}(t)\,dt, \qquad n = 1, 2, 3, \ldots, \tag{1}$$

with $Ki_0(x) = K_0(x)$, were first introduced by Bickley [1] in connection with the solution of heat convection problems. They arise in neutron transport calculations, and are widely used in nuclear reactor computer codes.

Taylor series and asymptotic expansions for $Ki_n(x)$ are developed in [2], [3] and [4], and [4] contains a discussion of the numerical stability of the four-term recurrence relation.

Chebyshev series and rational approximations to the $Ki_n(x)$ have been published in a number of reports. [5] gives 7S rational approximations to Ki_3 and Ki_4; [6] gives 5S, 7S and 8S rational approximations to Ki_1, Ki_2 and Ki_3, respectively; [7] gives 7S rational approximations to Ki_1; [8] gives 6S rational approximations to $Ki_1 - Ki_5$; [9] gives 20D Chebyshev series approximations to Ki_1; and [4] gives 12S rational approximations to Ki_1, Ki_2 and Ki_3 for $0 \leqslant x \leqslant 7$, and to Ki_{13}, Ki_{14} and Ki_{15} for $x \geqslant 7$. A number of the approximations in [5]–[9] suffer from significant digit cancellation.

This report gives rational minimax approximations to $Ki_1(x)$ for $x \geqslant 0$, to Ki_1, Ki_2 and Ki_3 for $0 \leqslant x \leqslant 6$, and to Ki_8, Ki_9 and Ki_{10} for $x \geqslant 6$, with relative errors ranging down to 10^{-23}. The approximations, combined with the recurrence relation, yield a stable method of computing $Ki_n(x)$, $n = 1, 2, \ldots, 10$, for the complete range of the argument. The results in [4] may be used to determine the accuracy of the Ki_n when the recurrence relation is extended to higher orders.

2. Functional Properties. Most of the results of this section are given in more general terms in [3].

Alternative definitions of $Ki_n(x)$ are

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$$Ki_n(x) = \int_0^\infty \frac{e^{-x\cosh t}}{\cosh^n t}\, dt \tag{2}$$

$$= \int_1^\infty \frac{e^{-xu}}{u^n(u^2-1)^{1/2}}\, du \tag{3}$$

$$= \int_0^{\pi/2} e^{-x\sec v} \cos^{n-1} v\, dv. \tag{4}$$

The derivative of Ki_n is given by

$$Ki_n'(x) = -Ki_{n-1}(x),$$

and higher derivatives may be computed recursively from the formulae

$$Ki_n^{(k)}(x) = -Ki_{n-1}^{(k-1)}(x), \qquad k = 1, 2, 3, \ldots, n = 0, 1, 2, \ldots,$$

$$Ki_{-1}^{(k)}(x) = -\frac{k}{x} Ki_{-1}^{(k-1)}(x) - Ki_0^{(k-1)}(x) - \frac{k-1}{x} Ki_0^{(k-2)}(x), \qquad k = 1, 2, 3, \ldots. \tag{5}$$

The latter equation is obtained by repeated differentiation of the formula

$$K_1'(x) = -\frac{1}{x} K_1(x) - K_0(x),$$

where K_0 and K_1 are the modified Bessel functions, and $Ki_0 \equiv K_0$, $Ki_{-1} \equiv K_1$.

By integrating (4) by parts we can derive the recurrence relation

$$(n-1)Ki_n(x) = x\,[Ki_{n-3}(x) - Ki_{n-1}(x)] + (n-2)Ki_{n-2}(x). \tag{6}$$

From (4) and (6) it follows that

$$Ki_n(0) = \begin{cases} \pi/2, & n = 1, \\ 1, & n = 2, \\ \dfrac{n-2}{n-1} Ki_{n-2}(0), & n \geqslant 3. \end{cases} \tag{7}$$

Ascending series for the Ki_n can be developed by repeated integration of the ascending series for $K_0(x)$. The resulting formula is

$$Ki_n(x) = P_n(x) + (-1)^n \left[\frac{x^n}{n!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma - \ln \frac{x}{2}\right) + \sum_{k=1}^{\infty} \frac{2^n (x/2)^{2k+n}}{(k!)^2 \prod_{j=1}^n (2k+j)} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{2k+1} + \cdots + \frac{1}{2k+n} - \gamma - \ln\frac{x}{2}\right)\right], \tag{8}$$

where γ is Euler's constant, and $P_n(x)$ is defined recursively, starting with $P_0(x) = 0$, by

$$P_n(x) = Ki_n(0) - \int_0^x P_{n-1}(t)\,dt, \quad n = 1, 2, 3, \ldots .$$

Asymptotic expansions for large arguments can be developed by writing (3) in the form

$$\text{(9)} \qquad \frac{e^{-x}}{(2x)^{1/2}} \int_0^\infty \frac{e^{-u}}{u^{1/2}} \left(1 + \frac{u}{x}\right)^{-n} \left(1 + \frac{u}{2x}\right)^{-1/2} du$$

and expanding the integrand binomially. The resulting asymptotic formula is

$$\text{(10)} \qquad Ki_n(x) \sim e^{-x} \left(\frac{\pi}{2x}\right)^{1/2} \sum_{m=0}^{\infty} (-1)^m a_m x^{-m}, \qquad x \longrightarrow \infty,$$

where

$$\text{(11)} \quad a_m = \begin{cases} 1, & m = 0, \\ \dfrac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m (n-1)!} \displaystyle\sum_{k=0}^{m} \frac{(2k)!(n+m-k-1)!}{8^k (k!)^2 (m-k)!}, & m = 1, 2, 3, \ldots . \end{cases}$$

The alternative formula

$$\text{(12)} \quad 2(m+1)a_{m+1} = (m + 1/2)\{(3m + 1/2 + 2n)a_m - (m - 1/2)(m - 1/2 + n)a_{m-1}\}$$

is derived in [4].

The change of variable $u = v^2$ in (9) gives the formula

$$\text{(13)} \qquad Ki_n(x) = (2/x)^{1/2} e^{-x} \int_0^\infty e^{-v^2} \left(1 + \frac{v^2}{x}\right)^{-n} \left(1 + \frac{v^2}{2x}\right)^{-1/2} dv,$$

which proves to be useful for computations.

3. **Stability of Recurrence Relation.** If (6) is used for forward recursion, the growth of the absolute error in $Ki_n(x)$ is determined by the factor $x/(n-1)$. Since Ki_n is a slowly decreasing function of n, the relative error is comparable to the absolute error, and forward recursion over a short range is stable provided $x/(n-1)$ is not large.

As a check of this result, $Ki_4, Ki_5, \ldots, Ki_{10}$ were computed by forward recursion on Ki_1, Ki_2 and Ki_3 for $x = 0(0.01)6.0$. The greatest loss of accuracy noted was one digit, which occurred near $x = 6$. In general, the accuracy loss was less than one digit.

If (6) is used for backward recursion, the main factor determining the growth of the absolute error is $(n-1)/x$. Since the relative error grows more slowly than the absolute error, backward recursion is stable provided $(n-1)/x$ is not large.

A numerical test consisted of computing $Ki_7, Ki_6, \ldots, Ki_1$ by backward recursion on Ki_8, Ki_9 and Ki_{10}, for $x \geqslant 6$. The greatest loss of accuracy noted was less than one digit, and occurred near $x = 6$. In general, the accuracy loss was very small.

[4] contains a more detailed discussion of the stability of the recurrence relation, and tables of values of the relative error from forward recursion for $n \leqslant 50$ and $x \leqslant 600$.

4. Generation of Approximations. Rational minimax approximations to $Ki_n(x)$ were computed in 29 decimal arithmetic on a CDC 6600 using a version of the second algorithm of Remes due to Ralston [10]. The relative error of the approximations was levelled to three digits.

The approximation forms and intervals are

$$\begin{aligned} Ki_n(x) &\simeq R_{lm}(x) + x^n \ln x S_{lm}(x^2), & 0 \leqslant x \leqslant 1, n = 1, 2, 3, \\ &\simeq x^{-1/2} e^{-x} R_{lm}(1/x), & x \geqslant 1, n = 1, \\ &\simeq x^{-1/2} e^{-x} R_{lm}(1/x), & 1 \leqslant x \leqslant 6, n = 1, 2, 3, \\ &\simeq x^{-1/2} e^{-x} R_{lm}(1/x), & x \geqslant 6, n = 8, 9, 10, \end{aligned}$$

where $R_{lm}(x)$ and $S_{lm}(x)$ are rational functions of degree l in the numerator and m in the denominator.

For the range $0 \leqslant x \leqslant 1$, R_{lm} is positive for $n = 1, 2, 3$, while S_{lm} is positive for $n = 1, 3$ and negative for $n = 2$, and so a loss of accuracy occurs for $n = 1, 3$ by subtraction. However, the amount of cancellation is small, being less than 1 bit for $n = 1$, and considerably less for $n = 3$.

The master routine for the range $0 \leqslant x \leqslant 1$ uses the power series expansions in (8). For sufficiently large values of x, for which terms in the asymptotic series in (10) become less than 10^{-30} in magnitude, the master routine uses the series in (10). For the intermediate range $Ki_n(x)$ is computed by a local Taylor series expansion of the form

$$Ki_n(x_0 + h) = Ki_n(x_0) + \sum_{m=1}^{N} \frac{h^m}{m!} Ki_n^{(m)}(x_0), \tag{14}$$

where $Ki_n(x_0)$ is the closest of a set of reference values, and where the derivatives $Ki_n^{(m)}(x_0)$ are computed by (5). The table of reference values is constructed by using (10) at an appropriate large value, and then using (14) repeatedly with negative values of h.

As a check of the master routine, the results of (8) and (14) were compared for a range of values of x between 0.6 and 1 and for $n = 0, 1, 2, \ldots, 10$. The relative difference was less than 5×10^{-27} in every case. The values of $Ki_0(x)$ were compared to those in [11], and showed agreement to at least 26 digits. An independent check is provided by (13), which was evaluated by a 136-point Gauss-Hermite integration formula. Agreement to 26 digits was obtained for $n = 1, 2, \ldots, 10$, and $x > x_n$, where x_n increases slowly with n. Typical values of x_n are $x_1 = 4$ and $x_{10} = 8$. The quadrature formula becomes progressively less accurate as x decreases.

As a result of the tests, we conclude that the master routine is accurate to at least 26 digits.

5. Results. The details of the approximations are given in Tables 1–245, in a format similar to that used in [12]. Tables 1–13 summarize the best approximations in the L_∞ Walsh arrays of the functions, and Tables 14–245 give the coefficients of selected approximations. Tables 14–245 are included in the microfiche section of this issue.

TABLE 1

$$Ki_1(x) \approx P_\ell(x)/Q_m(x) + x \ln x\, S(x^2)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[0,1]	1.49	1	0	14
	2.43	1	1	15
	3.83	3	0	16
	5.59	1	3	17
	6.55	3	2	18
	7.89	3	3	19
	10.09	1	6	20
	11.06	5	3	21
	12.99	7	2	22
	14.44	7	3	23
	16.43	9	2	24
	17.94	9	3	25
	20.02	9	4	26
	21.59	9	5	27
	23.77	11	4	28

TABLE 2

$$Ki_1(x) \approx R(x) + x \ln x\, P_\ell(x^2)/Q_m(x^2)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[0,1]	1.38	0	0	29
	3.41	1	0	30
	5.77	1	1	31
	8.44	2	1	32
	11.24	3	1	33
	14.17	4	1	34
	17.25	4	2	35
	20.47	5	2	36
	23.76	6	2	37

TABLE 3

$$Ki_2(x) \approx P_\ell(x)/Q_m(x) + x^2 \ln x\, S(x^2)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[0,1]	1.04	0	1	38
	2.25	2	0	39
	3.26	2	1	40
	4.83	4	0	41
	6.75	2	3	42
	8.24	2	4	43
	9.56	3	4	44
	11.49	4	4	45
	13.38	2	7	46
	14.41	8	2	47
	15.89	8	3	48
	17.91	10	2	49
	19.45	10	3	50
	21.61	10	4	51
	23.21	10	5	52

TABLE 4

$$Ki_2(x) \approx R(x) + x^2 \ln x\ P_\ell(x^2)/Q_m(x^2)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[0,1]	1.68	0	0	53
	4.05	0	1	54
	7.35	0	2	55
	9.21	2	1	56
	12.07	3	1	57
	15.13	0	5	58
	18.23	4	2	59
	21.48	5	2	60
	24.76	5	3	61

TABLE 5

$$Ki_3(x) \approx P_\ell(x)/Q_m(x) + x^3 \ln x\ S(x^2)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[0,1]	1.13	1	0	62
	1.76	1	1	63
	3.40	3	0	64
	4.47	3	1	65
	6.23	3	2	66
	8.55	3	3	67
	9.89	4	3	68
	11.31	4	4	69
	12.69	7	2	70
	14.84	3	7	71
	16.16	4	7	72
	17.63	9	3	73
	19.71	9	4	74
	21.28	9	5	75
	23.46	11	4	76

TABLE 6

$$Ki_3(x) \approx R(x) + x^3 \ln x\ P_\ell(x^2)/Q_m(x^2)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[0,1]	1.90	0	0	77
	4.64	0	1	78
	7.33	0	2	79
	9.83	2	1	80
	12.97	0	4	81
	15.84	3	2	82
	19.06	4	2	83
	22.35	5	2	84
	24.78	8	0	85

TABLE 7

$$Ki_1(x) \approx x^{-\frac{1}{2}} e^{-x} P_\ell(z)/Q_m(z), \quad z = 1/x$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
$[0,1]_z$	1.84	0	1	86
	2.81	1	1	87
	3.75	1	2	88
	4.65	2	2	89
	5.53	2	3	90
	6.38	3	3	91
	7.22	3	4	92
	8.05	4	4	93
	8.86	4	5	94
	9.66	5	5	95
	10.45	5	6	96
	11.24	6	6	97
	12.01	6	7	98
	12.78	7	7	99
	13.54	7	8	100
	14.29	8	8	101
	15.04	8	9	102
	15.78	9	9	103
	16.51	9	10	104
	17.24	10	10	105
	17.97	10	11	106
	18.69	11	11	107
	19.41	11	12	108
	20.12	12	12	109
	20.83	12	13	110
	21.53	13	13	111
	22.18	13	14	112

TABLE 8

$$Ki_1(x) \approx x^{-\frac{1}{2}} e^{-x} P_\ell(1/x)/Q_m(1/x)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[1,6]	2.13	0	1	113
	3.28	1	1	114
	4.41	1	2	115
	5.52	2	2	116
	6.62	2	3	117
	7.70	3	3	118
	8.78	3	4	119
	9.85	4	4	120
	10.91	4	5	121
	11.96	5	5	122
	13.01	5	6	123
	14.06	6	6	124
	15.10	6	7	125
	16.14	7	7	126
	17.17	7	8	127
	18.21	8	8	128
	19.23	8	9	129
	20.26	9	9	130
	21.29	9	10	131
	22.31	10	10	132
	23.33	10	11	133

TABLE 9

$$Ki_2(x) \approx x^{-1/2} e^{-x} P_\ell(1/x)/Q_m(1/x)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[1,6]	1.93	0	1	134
	3.04	1	1	135
	4.15	1	2	136
	5.23	2	2	137
	6.31	2	3	138
	7.38	3	3	139
	8.44	3	4	140
	9.50	4	4	141
	10.55	4	5	142
	11.59	5	5	143
	12.63	5	6	144
	13.67	6	6	145
	14.71	6	7	146
	15.74	7	7	147
	16.77	7	8	148
	17.79	8	8	149
	18.82	8	9	150
	19.84	9	9	151
	20.86	9	10	152
	21.88	10	10	153
	22.90	10	11	154

TABLE 10

$$Ki_3(x) \approx x^{-1/2} e^{-x} P_\ell(1/x)/Q_m(1/x)$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
[1,6]	1.81	0	1	155
	2.91	1	1	156
	3.99	1	2	157
	5.07	2	2	158
	6.13	2	3	159
	7.19	3	3	160
	8.24	3	4	161
	9.29	4	4	162
	10.33	4	5	163
	11.37	5	5	164
	12.40	5	6	165
	13.43	6	6	166
	14.46	6	7	167
	15.49	7	7	168
	16.51	7	8	169
	17.53	8	8	170
	18.55	8	9	171
	19.57	9	9	172
	20.59	9	10	173
	21.60	10	10	174
	22.62	10	11	175
	23.63	11	11	176

TABLE 11

$$Ki_8(x) \approx x^{-\frac{1}{2}} e^{-x} P_\ell(z)/Q_m(z), \quad z = 1/x$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
$[0,1/6]_z$	1.90	0	1	177
	3.04	1	1	178
	4.12	1	2	179
	5.20	2	2	180
	6.26	2	3	181
	7.29	3	3	182
	8.32	3	4	183
	9.33	4	4	184
	10.33	4	5	185
	11.31	5	5	186
	12.29	5	6	187
	13.26	6	6	188
	14.22	6	7	189
	15.17	7	7	190
	16.12	7	8	191
	17.05	8	8	192
	17.98	8	9	193
	18.91	9	9	194
	19.83	9	10	195
	20.74	10	10	196
	21.65	10	11	197
	22.55	11	11	198
	23.45	11	12	199

TABLE 12

$$Ki_9(x) \approx x^{-\frac{1}{2}} e^{-x} P_\ell(z)/Q_m(z), \quad z = 1/x$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
$[0,1/6]_z$	1.84	0	1	200
	2.96	1	1	201
	4.04	1	2	202
	5.09	2	2	203
	6.14	2	3	204
	7.16	3	3	205
	8.17	3	4	206
	9.17	4	4	207
	10.16	4	5	208
	11.14	5	5	209
	12.11	5	6	210
	13.07	6	6	211
	14.02	6	7	212
	14.96	7	7	213
	15.90	7	8	214
	16.83	8	8	215
	17.75	8	9	216
	18.67	9	9	217
	19.59	9	10	218
	20.49	10	10	219
	21.39	10	11	220
	22.29	11	11	221
	23.18	11	12	222

TABLE 13

$$Ki_{10}(x) \approx x^{-\frac{1}{2}} e^{-x} P_\ell(z)/Q_m(z), \quad z = 1/x$$

RANGE	PRECISION	ℓ	m	TABLE OF COEFFICIENTS
$[0,1/6]_z$	1.79	0	1	223
	2.90	1	1	224
	3.96	1	2	225
	5.00	2	2	226
	6.03	2	3	227
	7.04	3	3	228
	8.04	3	4	229
	9.03	4	4	230
	10.01	4	5	231
	10.98	5	5	232
	11.94	5	6	233
	12.89	6	6	234
	13.84	6	7	235
	14.77	7	7	236
	15.70	7	8	237
	16.63	8	8	238
	17.55	8	9	239
	18.46	9	9	240
	19.36	9	10	241
	20.26	10	10	242
	21.16	10	11	243
	22.05	11	11	244
	22.94	11	12	245

The precision is defined as

$$-\log_{10} \max_x \left| \frac{f(x) - R_{lm}(x)}{f(x)} \right|,$$

where $f(x)$ is the function being approximated and the maximum is taken over the appropriate interval.

For the range $0 \leqslant x \leqslant 1$ the first auxiliary function $R_{lm}(x)$ is ill-conditioned, and loses up to two significant digits by cancellation. To eliminate the cancellation the numerator and denominator were converted to minimal Newton form [13], and the resulting coefficients rounded off by an algorithm similar to that described in [12].

For the range $0 \leqslant x \leqslant 1$ the first auxiliary function $R_{1,2}$ for $Ki_2(x)$ is almost degenerate, and the rational function could only be found by Cody's method of artificial poles [14].

The approximations in Tables 14–245 were verified by comparing them with the master routine for 5000 pseudorandom values of the argument in each interval.

6. Use of Coefficients. The coefficients may be used to construct a subroutine to compute $Ki_n(x)$ for $n = 1, 2, \ldots, 10$, and for all values of x. For $x \leqslant 6$, approximations to Ki_1, Ki_2 and Ki_3, obtained from Tables 14–85 and 113–176, may be extended to Ki_4, $Ki_5, \ldots, Ki_{10}$ by forward recursion in (6) with the loss of at most one digit of accuracy. For $x \geqslant 6$, Tables 177–245 give Ki_8, Ki_9 and Ki_{10}, and these may be extended to Ki_7, $Ki_6, \ldots, Ki_1$ by backward recursion in (6) with less than one digit loss of accuracy.

Full range approximations to $Ki_1(x)$ are given in Tables 14–37 and 86–112, since it is the most commonly occurring function.

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<u>Table 220</u>

P00	(-6)	.30980	45026	86434	54052	78868	89
P01	(-4)	.32351	26784	53731	02745	47779	631
P02	(-2)	.13774	48359	33907	43013	48999	5520
P03	(-1)	.31889	94096	07821	36446	71858	7039
P04	(0)	.40530	16190	70136	99016	52363	3370
P05	(1)	.31225	49940	74137	92861	40737	4190
P06	(2)	.14004	40055	33348	24339	90218	3175
P07	(2)	.34758	62064	90829	46461	52985	5308
P08	(2)	.43233	95168	58088	81686	65485	856
P09	(2)	.22242	93457	04044	18609	01247	281
P10	(1)	.30044	76168	97981	86796	93466	1

Q00	(-6)	.24718	82295	60715	94364	05323	62
Q01	(-4)	.26955	82269	79396	89529	63561	697
Q02	(-2)	.12149	38345	89040	69653	81061	7477
Q03	(-1)	.29550	22858	98302	08467	37378	1861
Q04	(0)	.42498	92531	57229	61347	05641	8306
Q05	(1)	.37260	08178	28324	44737	95488	3935
Q06	(2)	.19819	68393	57298	73610	03270	6967
Q07	(2)	.61725	85420	67919	64337	76684	0587
Q08	(3)	.10448	75726	46710	87563	16750	1286
Q09	(2)	.83631	59142	54731	26682	16671	100
Q10	(2)	.23757	85710	27242	32574	16929	93
Q11	(1)	.10000	00000	00000	00000	00000	

Table 221

P00	(-7)	.18069	11916	42137	22617	66026	038
P01	(-5)	.20238	05211	46603	01411	25827	9258
P02	(-4)	.93299	10286	72493	53491	75826	6075
P03	(-2)	.23075	90087	93280	87986	70336	1636
P04	(-1)	.33483	88681	34839	19892	30076	4695
P05	(0)	.29319	17081	79713	95776	04373	9763
P06	(1)	.15382	88798	86045	92719	69054	6548
P07	(1)	.46581	39850	07571	80788	59763	4672
P08	(1)	.75522	25131	07101	38076	26362	3232
P09	(1)	.57089	84616	41547	43812	77238	3761
P10	(1)	.15144	12362	33949	84608	14976	627
P11	(-1)	.59059	36284	47914	48058	81054	

Q00	(-7)	.14417	07120	84333	03604	96727	347
Q01	(-5)	.16814	41886	84012	75345	95260	5184
Q02	(-4)	.81706	66377	89789	97963	24670	1745
Q03	(-2)	.21641	67756	25592	00266	96648	4129
Q04	(-1)	.34331	60481	95054	60265	36577	0386
Q05	(0)	.33764	88783	23821	23646	76740	6019
Q06	(1)	.20617	41042	37767	70262	05639	5107
Q07	(1)	.76190	68360	53387	61430	88541	1225
Q08	(2)	.16102	34821	86629	17701	14265	9245
Q09	(2)	.17544	03212	84772	91560	47882	5100
Q10	(1)	.80964	53895	55824	99765	26851	908
Q11	(1)	.10000	00000	00000	00000	00000	00

Table 222

P00	(-7)	.55203	00037	56298	04682	91887	402
P01	(-5)	.66117	89343	00917	57475	51159	8432
P02	(-3)	.32874	89448	83632	15855	14816	2768
P03	(-2)	.88643	48811	77075	83114	83620	6924
P04	(0)	.14217	01849	18183	41228	11103	4432
P05	(1)	.14011	03545	88487	36695	05318	2030
P06	(1)	.84790	63914	46076	58593	50836	9264
P07	(2)	.30657	74809	60477	85647	82624	2975
P08	(2)	.62518	73677	43893	75377	49226	0377
P09	(2)	.64838	29328	09012	58991	29288	5814
P10	(2)	.28150	13126	22065	68871	67959	5741
P11	(1)	.32421	66684	56129	49393	24466	206

Q00	(-7)	.44045	62170	97097	97974	53479	641
Q01	(-5)	.54791	55636	47534	96868	02564	2846
Q02	(-3)	.28608	08386	44466	42203	71640	8269
Q03	(-2)	.82159	26715	31252	01030	49558	6337
Q04	(0)	.14293	93823	25058	82283	28865	6131
Q05	(1)	.15647	09007	02670	88191	31880	8019
Q06	(2)	.10846	87654	59217	15019	46650	3358
Q07	(2)	.46781	59796	81549	26218	38903	9541
Q08	(3)	.12019	75920	73378	19141	10279	6461
Q09	(3)	.16996	70803	70108	16523	67552	2701
Q10	(3)	.11474	17652	44583	06320	55180	2557
Q11	(2)	.27719	57089	14178	28042	84145	7157
Q12	(1)	.10000	00000	00000	00000	00000	00

Table 223

P00	(0)	.32997 4
Q00	(0)	.26760 4
Q01	(1)	.10000 00

Table 224

P00	(0)	.18552 733
P01	(0)	.34781 66
Q00	(0)	.14821 738
Q01	(1)	.10000 000

Table 225

P00	(0)	.12748 7401
P01	(0)	.61119 942
Q00	(0)	.10173 1506
Q01	(1)	.10049 6855
Q02	(1)	.10000 0000

Table 226

P00	(-1)	.38682 82159
P01	(0)	.32525 74859
P02	(0)	.22108 2132
Q00	(-1)	.30864 09825
Q01	(0)	.41751 61305
Q02	(1)	.10000 00080

Table 227

P00	(-1)	.38989 46092 0
P01	(0)	.48351 21420 0
P02	(0)	.88100 89649
Q00	(-1)	.31109 11800 2
Q01	(0)	.54519 61104 7
Q02	(1)	.21567 63514 0
Q03	(1)	.10000 00000

Table 228

P00	(-2)	.83848 45088 0
P01	(0)	.13936 50868 800
P02	(0)	.45960 38551 22
P03	(0)	.16321 50243 0
Q00	(-2)	.66901 39045 1
Q01	(0)	.14548 34899 710
Q02	(0)	.82347 01489 28
Q03	(1)	.10000 00000 00

Table 229

P00	(-1)	.10847 19667 708
P01	(0)	.22898 98842 5245
P02	(1)	.11544 83841 2456
P03	(1)	.11400 88497 668
Q00	(-2)	.86548 10834 81
Q01	(0)	.22706 32100 3137
Q02	(1)	.17110 28525 3708
Q03	(1)	.36764 46108 768
Q04	(1)	.10000 00000 00

Table 230

P00	(-2)	.17686 47152 0268
P01	(-1)	.45779 76223 5167
P02	(0)	.32009 44670 50619
P03	(0)	.58803 56534 5628
P04	(0)	.13031 16708 396
Q00	(-2)	.14111 76257 4156
Q01	(-1)	.43759 24111 0497
Q02	(0)	.41870 92358 51586
Q03	(1)	.13458 34879 98129
Q04	(1)	.10000 00000 0000

Table 231

P00	(-2)	.27532 55321 54883
P01	(-1)	.85083 08969 81119
P02	(0)	.77459 30870 91180
P03	(1)	.22178 36224 52695 6
P04	(1)	.13923 47353 96654
Q00	(-2)	.21967 79913 22536
Q01	(-1)	.79144 98015 57501
Q02	(0)	.92876 35241 28374 8
Q03	(1)	.40775 90274 61963 6
Q04	(1)	.55455 60570 18914
Q05	(1)	.10000 00000 0000

Table 232

P00	(-3)	.36130 87887 07815 7
P01	(-1)	.13064 16736 02821 87
P02	(0)	.14846 68127 28078 59
P03	(0)	.59979 17379 51499 24
P04	(0)	.71282 82233 89445 6
P05	(0)	.10892 30566 78030
Q00	(-3)	.28828 27041 95356 5
Q01	(-1)	.11901 14628 59050 35
Q02	(0)	.16700 02958 22235 58
Q03	(0)	.94724 31210 84409 35
Q04	(1)	.19804 39424 91104 78
Q05	(1)	.10000 00000 00000 0

Table 233

P00	(-3)	.64831 72218 88680 62
P01	(-1)	.27005 36292 26774 919
P02	(0)	.37177 54026 16127 321
P03	(1)	.19871 81263 22314 593
P04	(1)	.37362 45297 15228 497
P05	(1)	.16382 26550 03208 77
Q00	(-3)	.51728 23018 48175 06
Q01	(-1)	.24198 23392 95662 719
Q02	(0)	.39830 56334 87430 162
Q03	(1)	.28092 69303 30919 590
Q04	(1)	.81908 67096 77212 283
Q05	(1)	.77369 98444 29281 84
Q06	(1)	.10000 00000 00000 0

Table 234

P00	(-4)	.70907 19732 42035 679
P01	(-2)	.33597 10932 27939 5802
P02	(-1)	.54760 51617 51758 4281
P03	(0)	.37025 24687 71619 6031
P04	(0)	.99418 10118 56672 0086
P05	(0)	.83466 44146 92294 274
P06	(-1)	.93858 12244 36844 8
Q00	(-4)	.56575 75799 47914 995
Q01	(-2)	.29706 12241 31629 1370
Q02	(-1)	.56473 15061 27760 9767
Q03	(0)	.48142 67812 79056 3944
Q04	(1)	.18406 32503 47737 18076
Q05	(1)	.27207 45940 96570 8088
Q06	(1)	.10000 00000 00000 000

Table 235

P00	(-3)	.14316 26554 60264 5717
P01	(-2)	.76350 66001 62632 11656
P02	(0)	.14472 27744 33474 25339 9
P03	(1)	.11981 01455 00669 46673 5
P04	(1)	.43814 55973 37875 45196
P05	(1)	.57720 32755 84989 8287
P06	(1)	.18790 43621 28994 1221
Q00	(-3)	.11422 72724 75286 7917
Q01	(-2)	.66773 16054 83533 05149
Q02	(0)	.14475 92293 75362 82000 6
Q03	(1)	.14596 81775 56454 98457 0
Q04	(1)	.78174 08184 38603 73185
Q05	(2)	.14677 05032 06949 23458
Q06	(2)	.10237 52145 88033 61755
Q07	(1)	.10000 00000 00000 000

Table 236

P00	(-4)	.13385 52399 82813 02540
P01	(-3)	.79646 10518 45903 89046 4
P02	(-1)	.17307 98018 60105 87322 60
P03	(0)	.17118 29816 60818 40447 19
P04	(0)	.78482 34105 08877 08865 1
P05	(1)	.15178 05989 83746 77824 5
P06	(0)	.95398 74579 29091 01822
P07	(-1)	.82652 38218 61853 463
Q00	(-4)	.10680 10293 84849 09280
Q01	(-3)	.69021 95040 98028 68973 0
Q02	(-1)	.16885 81474 38445 90000 15
Q03	(0)	.19801 01923 69006 69890 49
Q04	(1)	.11610 28916 62577 55966 66
Q05	(1)	.32219 21342 26682 97490 6
Q06	(1)	.35615 67698 25880 92481 7
Q07	(1)	.10000 00000 00000 00000

Table 237

P00	(-4)	.29848 57391 50359 30597 4
P01	(-2)	.19666 52996 66623 19690 95
P02	(-1)	.48424 93199 97643 88905 86
P03	(0)	.56126 90964 00778 53660 723
P04	(1)	.31767 20369 33803 50651 807
P05	(1)	.82798 46050 98904 31644 35
P06	(1)	.83804 71295 92486 17221 0
P07	(1)	.21154 57456 60943 83450
Q00	(-4)	.23815 71628 87903 11493 5
Q01	(-2)	.16912 17608 47669 14956 66
Q02	(-1)	.46276 26853 65185 73299 14
Q03	(0)	.62242 27393 84902 57261 833
Q04	(1)	.43487 67092 23522 73689 100
Q05	(2)	.15316 99714 43314 67767 902
Q06	(2)	.24214 77559 53358 99644 16
Q07	(2)	.13033 24507 92797 37687 0
Q08	(1)	.10000 00000 00000 00000

Table 238

P00	(-5)	.24332 60432 89816 50863 64
P01	(-3)	.17637 30047 23857 51316 777
P02	(-2)	.48726 56100 79838 83015 322
P03	(-1)	.65168 49434 72741 87096 6796
P04	(0)	.44376 98843 23159 08397 4780
P05	(1)	.14874 43788 92601 68260 2647
P06	(1)	.21845 38560 48521 99768 763
P07	(1)	.10712 04869 04760 35604 751
P08	(-1)	.73972 33657 13864 97331
Q00	(-5)	.19414 60931 82194 25239 04
Q01	(-3)	.15067 52846 87163 82163 801
Q02	(-2)	.45761 65895 29456 56186 690
Q03	(-1)	.69795 78054 06469 63640 5116
Q04	(0)	.57054 43111 59533 60798 1296
Q05	(1)	.24694 88191 07654 60478 0524
Q06	(1)	.52266 57643 06608 77832 954
Q07	(1)	.44992 04493 19676 39990 237
Q08	(1)	.10000 00000 00000 00000 00

Table 239

P00	(-5)	.59131 34150 83741 46268 01
P01	(-3)	.46884 28251 76173 24727 6555
P02	(-1)	.14412 52855 44927 21312 72638
P03	(0)	.21962 79642 78648 07729 15205
P04	(1)	.17634 14458 54390 81371 27629
P05	(1)	.73430 01810 98510 20526 99055
P06	(2)	.14623 62038 88271 32381 77331
P07	(2)	.11613 66327 77138 85542 6103
P08	(1)	.23480 05465 96266 97623 79

Q00	(-5)	.47179 98444 90933 47688 697
Q01	(-3)	.39826 21936 81425 78860 1371
Q02	(-1)	.13336 83234 11753 11300 11122
Q03	(0)	.22846 24168 22998 58585 41370
Q04	(1)	.21530 49601 59869 55533 37234
Q05	(2)	.11174 47316 26143 74110 80374 5
Q06	(2)	.38245 45645 36841 42917 41080
Q07	(2)	.37537 84592 93317 48534 0966
Q08	(2)	.16112 90257 39926 27459 481
Q09	(1)	.10000 00000 00000 00000 00

Table 240

P00	(-6)	.42661 62974 09568 21412 8440
P01	(-4)	.36815 58520 08531 08641 21872
P02	(-2)	.12503 10208 34144 71596 66163 2
P03	(-1)	.21481 09966 47756 39248 86619 1
P04	(0)	.20008 54611 55874 48390 09513 3
P05	(1)	.10080 48686 54278 09180 24275 38
P06	(1)	.25959 39100 56252 05327 60852 0
P07	(1)	.30071 57778 40441 60561 90414
P08	(1)	.11865 72895 79240 93644 6771
P09	(-1)	.67040 61599 77596 62495 17

Q00	(-6)	.34039 05570 89977 91987 6295
Q01	(-4)	.31119 88863 37692 87692 71299
Q02	(-2)	.11423 85264 21237 37685 02320 0
Q03	(-1)	.21799 77356 50028 75650 06326 0
Q04	(0)	.23398 66802 62938 00538 29603 9
Q05	(1)	.14283 85049 82933 37216 32074 38
Q06	(1)	.47811 76932 60740 34395 22188 4
Q07	(1)	.80003 17457 46232 05539 52197
Q08	(1)	.55301 35124 51010 04721 9297
Q09	(1)	.10000 00000 00000 00000 0000

Table 241

P00	(-5)	.11183 52000 28493 29914 50203
P01	(-3)	.10457 22805 92985 27126 38815 6
P02	(-2)	.38991 89865 94346 14816 65379 74
P03	(-1)	.74855 20744 74749 31837 02224 23
P04	(0)	.79815 11448 49858 93841 73093 49
P05	(1)	.47648 56109 10564 48800 01994 42
P06	(2)	.15319 93077 17686 97821 06570 49
P07	(2)	.24180 40298 32100 33045 09842 24
P08	(2)	.15521 22493 51895 26266 99497 6
P09	(1)	.25771 45406 07864 43896 7930

Q00	(-6)	.89231 57945 70349 71125 9175
Q01	(-4)	.88009 72662 02610 95327 72635
Q02	(-2)	.35236 09383 00896 91678 71571 98
Q03	(-1)	.74375 42392 24118 57063 98969 02
Q04	(0)	.89999 57136 16813 64992 21887 65
Q05	(1)	.63625 29823 78619 81264 56206 22
Q06	(2)	.25674 02103 34807 42285 32644 86
Q07	(2)	.55273 78422 63119 09828 41465 45
Q08	(2)	.55431 02764 68339 23206 67089 4
Q09	(2)	.19467 28114 34514 14801 76266
Q10	(1)	.10000 00000 00000 00000 000

Table 242

P00	(-7)	.72279 15785 49787 65576 89047
P01	(-5)	.72938 72139 80641 21905 85938 6
P02	(-3)	.29697 40311 33143 17693 94203 105
P03	(-2)	.63219 10472 27299 62805 07850 117
P04	(-1)	.76311 37522 62877 94756 91521 147
P05	(0)	.53083 61226 98283 85805 12623 749
P06	(1)	.20742 89656 50532 04597 31641 6831
P07	(1)	.42522 34801 27564 34406 82444 964
P08	(1)	.39978 47213 41739 20042 40980 32
P09	(1)	.13003 03190 97310 02698 43089 7
P10	(-1)	.61370 27227 81608 66312 620

Q00	(-7)	.57670 42412 83207 07975 91096
Q01	(-5)	.61152 28892 43833 85512 99565 4
Q02	(-3)	.26580 84504 82386 10998 44351 520
Q03	(-2)	.61676 14309 58079 27842 95524 811
Q04	(-1)	.83413 22379 15831 27389 28979 369
Q05	(0)	.67423 75156 03131 67679 51171 141
Q06	(1)	.32147 45143 32311 61194 41874 5744
Q07	(1)	.86041 45707 63387 13909 29746 094
Q08	(2)	.11698 43298 98431 88640 08489 086
Q09	(1)	.66514 25417 92638 65116 03687 2
Q10	(1)	.10000 00000 00000 00000 00000

Table 243

P00	(-6)	.20273	22098	98209	02184	26735	71
P01	(-4)	.21999	37667	91455	42121	18765	289
P02	(-3)	.97336	72927	06989	80017	91350	907
P03	(-1)	.22824	68944	56069	77582	15325	6438
P04	(0)	.30898	56474	81892	35841	30901	9113
P05	(1)	.24701	35076	56085	19199	08993	3556
P06	(2)	.11484	02949	99183	94896	46039	4217
P07	(2)	.29510	85363	18470	73684	87152	6203
P08	(2)	.37949	41083	05439	48436	04860	861
P09	(2)	.20149	99417	10047	24778	71862	237
P10	(1)	.28032	26500	57750	31189	25347	3

Q00	(-6)	.16175	69002	55226	81802	93265	07
Q01	(-4)	.18381	96711	33848	76878	69832	481
Q02	(-3)	.86385	51764	41932	28936	21569	561
Q03	(-1)	.21915	90148	13778	78079	51666	9619
Q04	(0)	.32880	62241	87052	40577	15483	7299
Q05	(1)	.30065	12878	93775	20503	45657	8697
Q06	(2)	.16666	77218	65291	32466	20541	7264
Q07	(2)	.54020	51169	06518	94789	70175	8188
Q08	(2)	.94967	92853	76043	02751	97885	598
Q09	(2)	.78723	45157	58502	60712	85970	216
Q10	(2)	.23088	08740	88414	15264	16174	30
Q11	(1)	.10000	00000	00000	00000	00000	

Table 244

P00	(-7)	.11854	49849	17900	94751	01242	421
P01	(-5)	.13787	82831	61175	68969	78972	1589
P02	(-4)	.66088	35151	03378	67561	03286	2241
P03	(-2)	.16951	20445	59854	43676	53297	0156
P04	(-1)	.25528	50303	74231	75523	01624	7833
P05	(0)	.23185	16286	26923	56885	05801	1296
P06	(1)	.12686	17095	96841	97998	04198	9905
P07	(1)	.39515	66724	77984	74490	37022	0747
P08	(1)	.66233	85369	62183	74420	37284	6466
P09	(1)	.51682	29936	83685	09637	74692	9516
P10	(1)	.14125	67678	66934	91974	80989	614
P11	(-1)	.56640	68787	82103	86385	48721	
Q00	(-8)	.94585	21322	66016	95131	59397	20
Q01	(-5)	.11485	84455	82171	18800	97210	7447
Q02	(-4)	.58144	97610	59131	33505	64813	0615
Q03	(-2)	.16050	52781	90354	73334	83702	7690
Q04	(-1)	.26540	46922	55453	76732	04669	6376
Q05	(0)	.27283	99652	81419	46431	62084	1538
Q06	(1)	.17302	46425	96927	56950	87380	3533
Q07	(1)	.66527	15997	49009	31069	22444	7221
Q08	(2)	.14603	14546	98017	65033	11539	0394
Q09	(2)	.16485	69143	30274	07861	49858	6550
Q10	(1)	.78605	26894	47159	70495	37570	629
Q11	(1)	.10000	00000	00000	00000	00000	00

Table 245

P00	(-7)	.35343	40142	78005	56899	49412	271
P01	(-5)	.43929	43125	59490	97025	77870	4529
P02	(-3)	.22668	23605	62431	20282	87348	2459
P03	(-2)	.63425	58346	87176	96068	28140	9730
P04	(0)	.10552	32823	42669	27020	03922	5120
P05	(1)	.10781	84553	10439	23436	31668	8267
P06	(1)	.67595	47635	18113	13217	25258	6491
P07	(2)	.25294	76152	41984	96766	79859	8406
P08	(2)	.53322	80004	12935	85092	35083	9842
P09	(2)	.57088	45149	14200	23843	99428	7220
P10	(2)	.25544	84217	98521	80504	67857	9617
P11	(1)	.30265	41651	31966	71518	29704	877

Q00	(-7)	.28199	95432	55000	11674	84513	904
Q01	(-5)	.36495	86262	31544	86719	99652	2102
Q02	(-3)	.19835	23794	53903	16162	21901	9261
Q03	(-2)	.59319	51453	45543	85825	99775	0349
Q04	(0)	.10749	14562	17072	92501	55987	9534
Q05	(1)	.12254	85471	76109	11777	77995	1537
Q06	(1)	.88439	47819	75090	58098	12299	9566
Q07	(2)	.39672	96366	26568	76731	78579	3816
Q08	(3)	.10586	67358	85881	05734	72978	0805
Q09	(3)	.15515	99358	71068	16319	49067	2470
Q10	(3)	.10828	76161	86850	15620	29603	3157
Q11	(2)	.26968	10569	14605	65915	28038	5309
Q12	(1)	.10000	00000	00000	00008	00000	00